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Categories of projective spaces

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Abstract

Starting with an abelian category $\mathscr{A}$, a natural construction produces a category $\mathbb{P}\mathscr{A}$ such that, when $\mathscr{A}$ is an abelian category of vector spaces, $\mathbb{P}\mathscr{A}$ is the corresponding category of *projective spaces*. The process of forming the category $\mathbb{P}\mathscr{A}$ destroys abelianess, but not completely, and the precise measure of what remains of it gives the possibility to reconstruct $\mathscr{A}$ out from $\mathbb{P}\mathscr{A}$, and allows to characterize categories of the form $\mathbb{P}\mathscr{A}$, for an abelian $\mathscr{A}$ (*projective categories*). The characterization is given in terms of the notion of "*Puppe exact category*" and of an appropriate notion of "*weak biproducts*". The proof of the characterization theorem relies on the theory of "*additive relations*".

0. Introduction

Starting with an abelian category $\mathscr{A}$, a natural construction described in [5] produces a category $\mathbb{P}\mathscr{A}$ such that, when $\mathscr{A}$ is an abelian category of vector spaces, $\mathbb{P}\mathscr{A}$ is the corresponding category of *projective spaces*. This construction is based on the subobject functor on $\mathscr{A}$, which in the linear context we call "*Grassmannian functor*". Clearly, categories of the form $\mathbb{P}\mathscr{A}$ for an abelian category $\mathscr{A}$ are to be thought of as "*categories of projective spaces*", and the natural question arises of their intrinsic characterization. Certainly, the process of forming the category $\mathbb{P}\mathscr{A}$ destroys abelianess, but not completely, and the precise measure of what remains of it gives the possibility to reconstruct $\mathscr{A}$ out from $\mathbb{P}\mathscr{A}$, thus giving us the solution to the characterization problem. This solution turns out to be given in terms of "*Puppe exact categories*" (see [8, 9]), a notion already exploited by the second named author in various papers as a frame for homological algebra more flexible than the one of abelian categories (see [6] and the references therein), and of a new notion of "*weak biproducts*". The

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proof of the characterization theorem relies on the theory of "*additive relations*", as developed e.g. in [3].

Since the paper [1] of the first author, where a characterization of categories of (possibly empty) "*affine spaces*" was given, the problem of having a similar understanding of categories of projective spaces was in order. This work provides the answer to this need, and makes possible a precise categorical analysis of the relations between the "additive-affine-projective" concepts.

We should mention that the discussions we had with Bill Lawvere during the preparation of this paper, when he was visiting Genoa Department, were extremely useful and illuminating. In particular, the idea to investigate the same construction, but with the slice (or "affine") functor instead of the Grassmannian functor, discussed in Section 6, is due to him.

1. Puppe exact categories

Definition 1. A *Puppe exact category* (or "p-exact" for short) is a well-powered category $\mathscr{A}$ with the following properties:

(1) $\mathscr{A}$ has a zero object;
(2) $\mathscr{A}$ has kernels and cokernels;
(3) every mono is a kernel and every epi is a cokernel.
(4) every arrow has an epi-mono factorization.

An exact functor between p-exact categories is a functor which preserves kernels and cokernels (and hence the zero object).

The reader will recognize in (1), (2) and (3) of the above definition the part of the definition of an abelian category, as given for instance in [4], not involving the biproducts. Property (4) for an abelian category follows from the existence of biproducts, and our reason to consider the four properties in the present paper is, it is the part of the notion of an abelian category which is stable under a very basic construction, first considered in this generality in [5], *the construction of the category* $\mathbb{P}\mathscr{A}$ *of the projective spaces of* $\mathscr{A}$, which we will call *the projective category of* $\mathscr{A}$. Before recalling such a construction, let us recall some basic properties of p-exact categories, which are needed in the following.

Lemma 2. *Let* $\mathscr{A}$ *be a p-exact category; then:*

(i) *for any arrow* f, *direct images* $f_\bullet(x)$ *and inverse images* $f^\bullet(y)$ *of subobjects exist*;

(ii) *for each object, the ordered sets of subobjects and of quotient objects are* modular lattices, *and they are anti-isomorphic via kernels and cokernels*;

(iii) *the operators* $f_\bullet$ *and* $f^\bullet$ *are adjoint, and* $f_\bullet \dashv f^\bullet$ *is a* "modular connection", *i.e. satisfies the* "Frobenius reciprocity law" *and its dual*

$$f_\bullet(x \cap f^\bullet(y)) = f_\bullet(x) \cap y, \qquad f^\bullet(f_\bullet(x) \cup y) = x \cup f^\bullet(y),$$

or, equivalently, the following equations:

$$f_\bullet f^\bullet(y) = y \cap f_\bullet(1), \qquad f^\bullet f_\bullet(x) = x \cup f^\bullet(0);$$

(iv) *given a commutative diagram*

$$\begin{array}{ccccc} X & \xrightarrow{k} & Y & \xrightarrow{q} & Z \\ {\scriptstyle u}\downarrow & & \downarrow{\scriptstyle n} & & \downarrow{\scriptstyle m} \\ U & \xrightarrow[h]{} & V & \xrightarrow[p]{} & W \end{array}$$

where the rows are short exact and m,n *are monos, than the right-hand square is a pullback if and only if* u *is an iso.*

Observe that from point (ii) it follows that the dual of a p-exact category is again p-exact. We refer to [5] and to the references therein for the proof of this lemma, as well as for other properties of p-exact categories and for more examples of p-exact categories other than the ones discussed in this paper. In particular, in view of what will be discussed in Section 3, we should point out that the notion of a p-exact category supports an appropriate notion of *relation* and of *composition* of relations, allowing the formation of the category of relations, and we refer again to [5] for a survey on this matter, and for the references to the literature on the subject of relations in p-exact categories.

Observe that the properties of p-exact categories allow us to consider the *Grassmannian functor*

$$\mathscr{G} : \mathscr{A} \longrightarrow \text{Poset}$$

given by the poset of *subobjects* $\mathscr{G}A = \text{Sub}(A)$ on objects and by the direct image operator $\mathscr{G}f = f_\bullet$ on arrows.

The basic fact studied in [5] is the following: let $\mathscr{A}$ be a p-exact category, and let

$$P : \mathscr{A} \longrightarrow \mathbb{P}\mathscr{A}$$

be the *full image of the Grassmannian functor*; so, $\mathbb{P}\mathscr{A}$ has the same objects of $\mathscr{A}$ and $\mathbb{P}\mathscr{A}(A,B)$ is the image under $\mathscr{G}$ of $\mathscr{A}(A,B)$, or equivalently the quotient of $\mathscr{A}(A,B)$ under the equivalence relation

$$f \simeq g \quad \text{if and only if } f_\bullet = g_\bullet, \quad \text{if and only if } f^\bullet = g^\bullet.$$

Theorem 3 (see Grandis [5]). *Let* $\mathscr{A}$ *be a p-exact category*; *then*

(i) $\mathbb{P}\mathscr{A}$ *is a p-exact category and the quotient functor* $\mathscr{A} \longrightarrow \mathbb{P}\mathscr{A}$ *preserves and reflects kernels, cokernels and zero*;

(ii) *the lattice of subobjects of each object in* $\mathscr{A}$ *is the same as in* $\mathbb{P}\mathscr{A}$ *and* $\mathbb{P}\mathscr{A} \simeq \mathbb{P}\,\mathbb{P}\mathscr{A}$;

(iii) *when $\mathscr{A}$ is the p-exact category of vector spaces over a field K, then $\mathbb{P}\mathscr{A}$ is equivalent to the category of the projective spaces over K.*

As for the last sentence, the point is that in the case of the category of vector spaces, one can show that two linear functions f and g induce the same function on the Grassmannians if and only if there exists an invertible scalar k such that $kf = g$. In the case of general rings, this is not anymore true. For instance, for abelian groups, the cyclic group $\mathbf{Z}/p$ with p a prime, has $p-1$ automorphisms which are all Grassmannian equivalents, since $\mathbf{Z}/p$ is simple, but cannot be all related through the invertible scalars of $\mathbf{Z}$ when $p \geq 5$. Thus our definition of the category of projective spaces differs from the one defined using invertibles of the ring. Observe that we do not include the "dimension" in our definition of the Grassmannian, the only linear functions inducing a dimension preserving function on the Grassmannians being the isomorphisms, at least in the case of vector spaces.

The above theorem seems to motivate an abstract definition of a category of projective spaces, or "*projective category*", as that of a *p-exact category for which the Grassmannian functor is faithful*. A quite surprising example in this sense, studied in [5], is the category **Mlc** of "*modular lattices and modular connections*", so that **Mlc** appears as a universal recipient for projective categories, i.e. any projective category has a faithful exact functor in it, namely the Grassmannian functor. This definition is certainly part of the game, but cannot be the whole story, since it is a common experience that, when we start from the *abelian* category of vector spaces, we can reconstruct it out from the category of projective spaces. So, the question is: if $\mathscr{A}$ is an *abelian* category, can we reconstruct $\mathscr{A}$ out from the projective category $\mathbb{P}\mathscr{A}$? Furthermore, can we *characterize* the categories of the form $\mathbb{P}\mathscr{A}$ for an abelian category $\mathscr{A}$? Such a characterization would then give us the definition of a (Grassmannian) projective category.

2. Projective categories

Recalling that a p-exact category is *abelian* if and only if it has *biproducts*, and actually, products only are enough, the subtraction being the cokernel of the diagonal (see [7]), we need to investigate what remains of the biproducts in $\mathscr{A}$ after forming the projective category $\mathbb{P}\mathscr{A}$. Let us first recall the characterization of the equivalence relation on the hom-sets of $\mathscr{A}$ which gives the hom-sets of $\mathbb{P}\mathscr{A}$ (see [5]):

Two parallel arrows $f,g: A \longrightarrow B$ are equivalent if and only if, considering the epi–mono factorizations $f = m\,p$ and $g = n\,q$, there are (unique) isomorphisms $a, b: I \longrightarrow J$, where I and J are the codomains of p and q, respectively, such that $a p = q$, $nb = m$ and $a^{\bullet} = b^{\bullet}$.

Observe that the composites $\theta = b^{-1}a$ and $\phi = b\,a^{-1}$ are automorphisms of I and J which are equivalent to the identity, and such that $f = n\phi q$ and $g = m\theta p$. It is now immediate to show that $\mathbb{P}\mathscr{A}$ inherits the following weak form of biproducts:

first of all observe that since the projection functor P is the identity on objects, the image under P of the projections, injections, diagonals, codiagonals, associativity and symmetry isomorphisms satisfy in $\mathbb{P}\mathscr{A}$ all equational properties satisfied in $\mathscr{A}$. In other words, recalling that the *free additive category*

$$\mathrm{Matr}\,(\mathscr{A}_0)$$

on the "set" $\mathscr{A}_0$ of the objects of $\mathscr{A}$, is the one having as *objects* finite families $(A_i)_{i\in I}$ of objects of $\mathscr{A}$ and as *arrows* $M:(A_i)_{i\in I} \longrightarrow (B_j)_{j\in J}$ matrices $M=(m_{ji})$ of integers with the property that when $A_i \neq B_j$, $m_{ji}=0$, then P induces a functor

$$\oplus : \mathrm{Matr}\,(\mathscr{A}_0) \longrightarrow \mathbb{P}\mathscr{A}\,,$$

which *preserves the zero object*, and which satisfies the following properties:

For each pair of objects A, B of $\mathscr{A}$, consistently denoting with the same name the image under $\oplus$ of the arrows of Matr $(\mathscr{A}_0)$,

$$A \underset{p_A}{\overset{i_A}{\rightleftarrows}} A\oplus B \underset{p_B}{\overset{i_B}{\leftrightarrows}} B\,,$$

then

(1) *$\oplus$ restricted to the one element families is the inclusion of the objects of $\mathscr{A}$;*

(2) *the following diagram is a short exact sequence:*

$$A \xrightarrow{i_A} A\oplus B \xrightarrow{p_B} B\,;$$

(3) *the weak universal property holds for projections: for each pair of arrows*

$$f:U\longrightarrow A, \quad g:U\longrightarrow B\,,$$

there exists an arrow

$$\langle f,g\rangle : U \longrightarrow A\oplus B$$

such that $p_A\langle f,g\rangle = f$ and $p_B\langle f,g\rangle = g$;

(4) *for each pair of arrows $f:X\longrightarrow A$, $g:Y\longrightarrow B$, there is an arrow*

$$\gamma=\gamma_{fg} : X\oplus Y\oplus X \longrightarrow A\oplus B\oplus A$$

of diagonal components (f,g,f) satisfying the following equations:

$$S_{13}\,\gamma=\gamma\, S_{13}, \quad P_{12}\,\gamma = \gamma\, P_{12}, \quad P_{13}\,\gamma=\gamma\, P_{13}, \quad D\gamma=\gamma D\,,$$

where S, P and D are the matrices *of* Matr $(\mathscr{A}_0)$ *defined by:*

$$S_{13}(x,y,z) = (z,y,x), \qquad P_{12}(x,y,z)=(x,y,0),$$
$$P_{13}(x,y,z) = (x,0,z), \qquad D(x,y,z)=(x,0,x)\,.$$

Here and in the following, whenever we say that a morphism

$$h: \bigoplus_r X_r \longrightarrow \bigoplus_r Y_r$$

has "diagonal components" $h_r = p_r h i_r$, it is understood that all other components $h_r s = p_r h i_s$ are zero, for $r \neq s$.

Definition 4. A structure of *projective biproducts* on a p-exact category $\mathscr{P}$ is given by a zero preserving functor

$$\oplus : \mathrm{Matr}(\mathscr{P}_0) \longrightarrow \mathscr{P},$$

satisfying all the properties listed above.

Clearly any abelian category always has a canonical structure of projective biproducts, but the Grassmannian functor is almost never faithful (e.g. for vector spaces it is faithful if and only if the ground field is $\mathbf{Z}/2$); thus:

Definition 5. A (Grassmannian) *projective category* is a category $\mathscr{P}$ such that:
(1) $\mathscr{P}$ is a p-exact category
(2) $\mathscr{P}$ is equipped with a structure of projective biproducts;
(3) the Grassmannian functor is faithful.

Our goal is to show that any projective category $\mathscr{P}$ arises as the projective category of an abelian category; and we shall be able to measure the ambiguity in axiom (2): we will show that, for each pair of structures of projective biproducts $\oplus$ and $\oplus'$ on $\mathscr{P}$, there exists an isomorphism $\oplus \simeq \oplus'$ commuting with projections (and injections); however, this isomorphism *need not be natural*, and we can only show that it induces a *graph* morphism between the two abelian categories constructed out from the two structures of projective biproducts – composition and identities not necessarily being preserved, due to the lack of naturality.

3. Additive relations

Observe that when $\mathscr{P}$ is a projective category of the form $\mathbb{P}\mathscr{A}$ for an abelian category A, the arrows $X \longrightarrow Y$ in $\mathscr{A}$ are recovered in $\mathscr{P}$ as *subobjects* of $X \oplus Y$ for which the composite with the first projection is an isomorphism. Hence, starting with a projective category $\mathscr{P}$ and looking for an abelian category $\mathbb{A}\mathscr{P}$ whose associated projective category is (equivalent to) $\mathscr{P}$, we *must* define an arrow

$$X \overset{R}{\dashrightarrow} Y$$

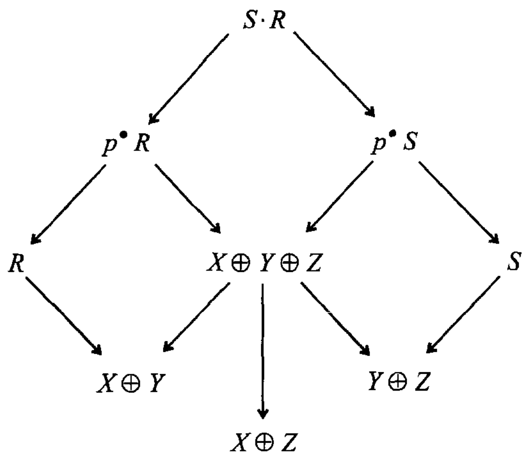
in $\mathbb{A}\mathscr{P}$ as a *subobject* ("*additive relation*"):

$$R \hookrightarrow X \oplus Y,$$

of $\mathscr{P}$ such that the composite with the first projection is an isomorphism. But now, what about *composition*? In the models $\mathscr{P} = \mathbb{P}\mathscr{A}$, one can easily check that the *relational* *composition* is the one which gives back the composition in $\mathscr{A}$, where the relational composition SR of

$$X \overset{R}{\dashrightarrow} Y \overset{S}{\dashrightarrow} Z$$

is defined by mean of the canonical structure of projective biproducts, first considering the diagram



where the arrows between biproducts are the appropriate projections, and where all the squares are inverse images, and then taking the image of the arrow $S \cdot R \longrightarrow X \oplus Z$. Ignoring the proper names of the projections involved, we can summarize the above definition in the formula:

$$SR = p_\bullet(p^\bullet R \cap p^\bullet S).$$

Certainly, we can make the same definition for any projective category $\mathscr{P}$, using the structure of projective biproducts; then, using the properties of the direct and inverse images in a p-exact category, basically the *Frobenius reciprocity laws*, which we already mentioned, and the *Beck–Chevalley condition* for projections, which we will show to hold in any projective category, we can still prove that the composition so defined is *associative*. Yet, what about *identities*? Using the structure of projective products, the image under the functor $\oplus$ of the diagonal d_X on X in $\mathrm{Matr}(\mathscr{P}_0)$ is still a mono in $\mathscr{P}$, which we will denote by the same letter,

$$d_X : X \longrightarrow X \oplus X,$$

and hence determines an endorelation 1_X on X in $\mathscr{P}$, which does the job. So, we will show that starting with a projective category $\mathscr{P}$, we can define a category, in fact a locally ordered bicategory, of "*additive relations*" on $\mathscr{P}$, which we will denote by

$$\mathbb{A}\,\mathrm{Rel}(\mathscr{P}).$$

As for any locally ordered bicategory, we can take the subcategory of the *relations* R *with right adjoint* R^*, i.e. with a relation R^* such that $RR^* \leq 1$ and $1 \leq R^*R$, also called "*maps*"

$$\mathbb{A}\mathscr{P} = \mathrm{Map}(\mathbb{A}\mathrm{Rel}(\mathscr{P})),$$

and our aim is to show that $\mathbb{A}\mathscr{P}$ is an *abelian* category such that $\mathbb{P}\mathbb{A}\mathscr{P}$ is equivalent to $\mathscr{P}$.

The cheapest way to prove that $\mathbb{A}\mathscr{P}$ is an abelian category, is to show that $\mathbb{A}\,\mathrm{Rel}\,(\mathscr{P})$ satisfies all the axioms for the bicategory of relations of an abelian category, which is then determined as the category of maps. Such an axiomatic has been provided by various authors, and we will follow here the one contained in [3]. We now recall the axioms, referring to [3] for the proof of the characterization. We will give a more convenient set of axioms than the ones in [3], modified as follows on the basis of [2, Section 7].

Definition 6. An *abelian bicategory* is a symmetric monoidal category

$$(\mathscr{B}, \oplus, 0)$$

enriched over ordered sets as a monoidal category, equipped with two lax natural transformations

$$\Delta : Id \Rightarrow Id \oplus Id, \qquad \top : Id \Rightarrow 0$$

and with two op-lax natural transformations

$$\nabla : Id \oplus Id \Rightarrow Id, \qquad \perp : 0 \Rightarrow Id,$$

where Id denotes the identity functor and 0 the constant functor at 0, satisfying the following axioms:

(1) The components Δ_X and $\top_X$ satisfy the equations for a cocommutative coalgebra, and ∇_X and $\perp_X$ the ones for a commutative algebra.

(2) $\top_0 = 1_0 = \perp_0$.

(3) Each of the four components Δ_X, $\top_X$, ∇_X and $\perp_X$ as a right adjoint, which we will denote by the same letter upperstarred.

(4) the inequalities given by coassociativity and associativity by mean of adjointness are in fact equalities:

$$\Delta\,\Delta^* = (\Delta^* \oplus 1)(1 \oplus \Delta), \qquad \nabla^*\,\nabla = (\nabla \oplus 1)(1 \oplus \nabla^*).$$

(5) Reflexive and coreflexive endomorphisms spilt.

4. Reconstructing an Abelian category

In [1], it is proved that the axioms of an abelian bicategory *characterize* the bicategories of additive relations of abelian categories, so that the first part of our problem reduces to showing that $\mathbb{A}\,\mathrm{Rel}(\mathscr{P})$ is an abelian bicategory. We begin developing some consequences of the axiom of projective biproducts on a p-exact category.

Lemma 7. *Let $\mathscr{P}$ be a p-exact category equipped with projective biproducts:*

(i) *an arrow $U \longrightarrow X \oplus Y$ is uniquely determined by the components when one of them is zero;*

(ii) *given two morphisms $f_r : X_r \longrightarrow Y_r$, $r = 1, 2$, a morphism*

$$f : X_1 \oplus X_2 \longrightarrow Y_1 \oplus Y_2$$

is consistent with them with respect to the projections (i.e. $p_r f = f_r p_r$, $r = 1,2$) if and only if it is consistent with them with respect to the injections (i.e. $f i_r = i_r f_r$, $r = 1,2$), if and only if it has diagonal components f_r (i.e. $p_r f i_r = f_r$ and $p_r f i_s = 0$, when $r \neq s$); if the components f_r are both mono or epi, then so is any such f;

(iii) *the dual property of (3) of projective biproducts holds for injections;*

(iv) *given a monomorphism $m : X \longrightarrow Y$, then for any object U and any arrow k of diagonal components $(1_U, m)$ the following commutative square is a pullback:*

$$\begin{array}{ccc} U \oplus X & \xrightarrow{\;k\;} & U \oplus Y \\ \downarrow{\scriptstyle p_X} & & \downarrow{\scriptstyle p_Y} \\ X & \xrightarrow[\;m\;]{} & Y \end{array}$$

The dual statement holds for injections and an epimorphism.

(v) *Beck–Chevalley condition holds for projections: for any commutative square*

$$\begin{array}{ccc} X \oplus Y \oplus Z & \xrightarrow{\;r\;} & Y \oplus Z \\ \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle q} \\ X \oplus Y & \xrightarrow[\;s\;]{} & Z \end{array}$$

where the arrows involved are projections, the square

$$\begin{array}{ccc} \mathscr{G}(X \oplus Y \oplus Z) & \xleftarrow{\;r^{\bullet}\;} & \mathscr{G}(Y \oplus Z) \\ \downarrow{\scriptstyle p_{\bullet}} & & \downarrow{\scriptstyle q_{\bullet}} \\ \mathscr{G}(X \otimes Y) & \xleftarrow[\;s^{\bullet}\;]{} & \mathscr{G}(Z) \end{array}$$

commutes.

Proof. Properties (i) and (ii) follow quite easily from the exactness condition (2) in Definition 6. In particular, property (ii) can be proved as follows: given f with $p_r f = f_r p_r$ (and such an arrow exists by the weak universal property of the projections), we have $p_2 f i_1 = f p_2 i_1 = 0$; by the exactness condition (2) of projective biproducts, $f i_1$ factors as $i_1 h$ and h is actually f_1 ($h = p_1 i_1 h = p_1 f i_1 = f_1 p_1 i_1 = f_1$); similarly, $f i_2 = i_2 f_2$. Dually, the consistency with injections implies the one with projections. Finally, if f_1 and f_2 are mono, consider an arrow x with $f x = 0$ and deduce that $x = 0$ by point (i). Property (iii) follows immediately from (ii), using the codiagonal $\delta : X \oplus X \longrightarrow X$. Property (iv) follows from Lemma 2 and the last assertion in property (ii), and Beck–Chevalley condition for projections is proved as follows:

$$\begin{aligned} s^\bullet q_\bullet(A) &= p_\bullet p^\bullet s^\bullet q_\bullet(A) = p_\bullet r^\bullet q^\bullet q_\bullet(A) = p_\bullet r^\bullet(A \cup q^\bullet(0)) \\ &= p_\bullet r^\bullet(A \cup r_\bullet r^\bullet q^\bullet(0)) = p_\bullet(r^\bullet(A) \cup r^\bullet q^\bullet(0)) \\ &= p_\bullet r^\bullet(A) \cup p_\bullet p^\bullet s^\bullet(0) \\ &= p_\bullet r^\bullet(A) \cup s^\bullet(0) = p_\bullet r^\bullet(A), \end{aligned}$$

where this last equation holds because

$$s^\bullet(0) = s^\bullet j^\bullet(0) = i^\bullet r^\bullet(0) \leq i^\bullet p^\bullet p_\bullet r^\bullet(0) = p_\bullet r^\bullet(0),$$

i and j denoting the appropriate injections. □

Observe that property (ii) implies that the notion of projective biproducts on a p-exact category is a selfdual notion. We can now prove the main result of this section:

Theorem 8. *Let $\mathscr{P}$ be a p-exact category with projective biproducts; then the order enriched category $\mathbb{A}\,\mathrm{Rel}(\mathscr{P})$ is an abelian bicategory, so that $\mathbb{A}\mathscr{P}$ is an abelian category.*

Proof. We already mentioned how to define composition and identities; the proof of the associativity and identity axioms are now a straightforward consequence of the properties stated in the previous corollary, basically of the Beck–Chevalley condition and of the Frobenius reciprocity laws.

Let us now define the tensor product in $\mathscr{B} = \mathbb{A}\,\mathrm{Rel}(\mathscr{P})$: given two additive relations

$$X \overset{R}{\dashrightarrow} Y, \qquad A \overset{T}{\dashrightarrow} B,$$

define

$$X \oplus A \overset{R \oplus T}{\dashrightarrow} Y \oplus B,$$

as the additive relation

$$p^\bullet_{XY} R \cap p^\bullet_{AB} T \hookrightarrow X \oplus A \oplus Y \oplus B.$$

Using the Beck–Chevalley condition and the Frobenius reciprocity laws, it is again a straightforward checking to show that this definition gives a symmetric "tensor product" on $\mathbb{A}\,\mathrm{Rel}(\mathscr{P})$, i.e. that it is functorial, that the symmetry and associativity isomorphisms of the projective biproducts structure provide the coherent isomorphisms for a symmetric tensor product, and that the object 0 is the *unit* of this tensor product.

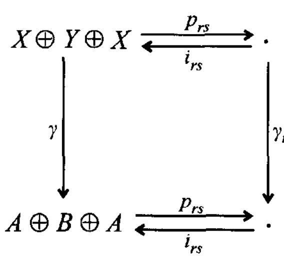
As for the four transformations Δ, $\top$, ∇ and $\bot$, we can take in $\mathrm{Matr}(\mathscr{P}_0)$ the *graphs* of the appropriate arrows (diagonal, terminal, codiagonal, initial), and then consider their images under $\oplus$, which still are mono; we must show that they are *maps*, i.e. that they have a *right adjoint*; but one can easily check that for graphs of arrows in $\mathrm{Matr}(\mathscr{P}_0)$, the adjoints are provided by the composites of the graphs with the appropriate symmetry isomorphisms. The required lax and op-lax naturalities are also easy, as well as axiom (2) of the definition 6.

The first of the two equations of axiom (4) in Definition 6 is easily proved using property (iv) of Lemma 7; as for the second equation, one should use again property (iv) of Lemma 7, and for the first time the *subtraction* arrow of $\mathrm{Matr}(\mathscr{P}_0)$.

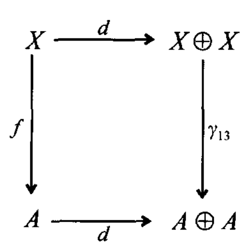
Before proving axiom (5), let us first point out that we can calculate the *opposite* $R^\circ : Y \to X$ of an additive relation $R : X \to Y$, as defined in [3]

$$Y \xrightarrow{\eta_X \oplus 1_Y} X \oplus X \oplus Y \xrightarrow{1_X \oplus R \oplus 1_Y} X \oplus Y \oplus Y \xrightarrow{1_X \oplus \varepsilon_Y} X,$$

η and ε being the additive relations $\eta = \Delta \top^*$ and $\varepsilon = \top \Delta^*$, to be the composite of R with the symmetry isomorphism. Then, with a bit of effort, we can show that *maps* in $\mathbb{A}\,\mathrm{Rel}(\mathscr{P})$ are precisely the additive relations $R : X \to Y$ whose composition with the first projection is an isomorphism; for the proof the lemma contained in [3] may be useful, showing that maps are precisely coalgebra homomorphisms, the *adjoint* being then necessarily the *opposite* relation. Now, using the same notations as in axiom (4) of Definition 6, and noting that the morphism γ also commutes with $P_{23} = S_{13} P_{12} S_{13}$, define $\gamma_{rs} = p_{rs}\,\gamma\, i_{rs}$ for $r, s = 1, 2, 3$, and note that the following squares commute:



i.e. "γ restricts to the three coordinate hyperplanes"; also note that the following square commutes:



d being the diagonal arrow of Matr$(\mathscr{P}_0)$. Finally, recall that, as discussed in Section 3 of [3], coreflexive relations split if and only if for each additive relation

$$Y \overset{R}{\dashrightarrow} 0$$

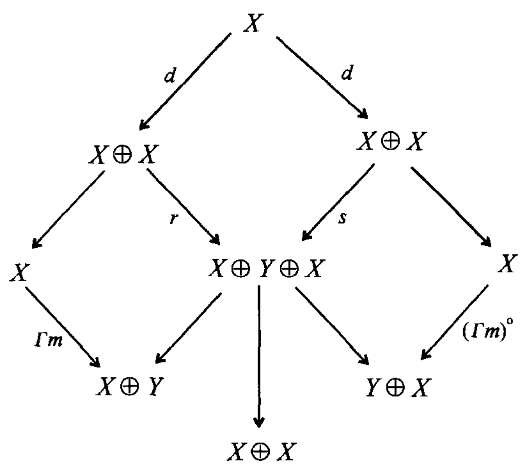
there exists a *map* $M : X \to Y$ *in* Map$(\mathbb{A}\mathrm{Rel}(\mathscr{P}))$ such that $M^* M = M^\circ M = 1$ and $\top M^\circ = R$. Now, to show the first part of axiom (5), first note that an additive relation $R : Y \to 0$ is nothing but a *monomorphism* $m : X \rightarrowtail Y$ in $\mathscr{P}$, and the point is that, using axiom (4) of Definition 6, we can show that a monomorphism $m : X \rightarrowtail Y$ in $\mathscr{P}$ has a convenient notion of a *graph*, as follows: first consider an arrow γ_{1m} associated to 1_X and m by axiom (4) of projective biproducts; define a *graph* of m as the composite

$$\Gamma m = p_{12}\, \gamma_{1m}\, d_3\,,$$

d_3 being the composite $(1 \oplus d)d = (d \oplus 1)d$. Γm is a map, since the composition with the first projection is the identity; hence the adjoint $(\Gamma m)^*$ is the opposite relation $(\Gamma m)^\circ = \sigma \Gamma m$. Using the equations in axiom (4) we can show that

$$(\Gamma m)^\circ = p_{23}\, \gamma_{1m}\, d_3\,.$$

The equation $\top (\Gamma m)^\circ = R$ is rather simple to prove. As for the equation $(\Gamma m)^\circ \Gamma m = 1$, consider the diagram



where the unnamed arrows are the appropriate projections, and where $r = \gamma_{1m}(d \oplus 1_X)$ and $s = \gamma_{1m}(1_X \oplus d)$. By point (iv) of Lemma 7 and the equations of γ_{1m}, the two bottom squares are pullbacks; the top square is also a pullback, since γ_{1m} is mono, because m is a mono, and $(d \oplus 1_X)d = (1_X \oplus d)d$ is a pullback in $\mathscr{P}$ (indeed, composing with the square $p_{12}(d \oplus 1) = d\, p_1$, which is a pullback by point (iv) of Lemma 7, one has a trivial pullback). Finally, the composite $p_{1,3}\, s d$ is d, so that the relational composition $(\Gamma m)^\circ \Gamma m$ is the identity. Hence coreflexive additive relations split.

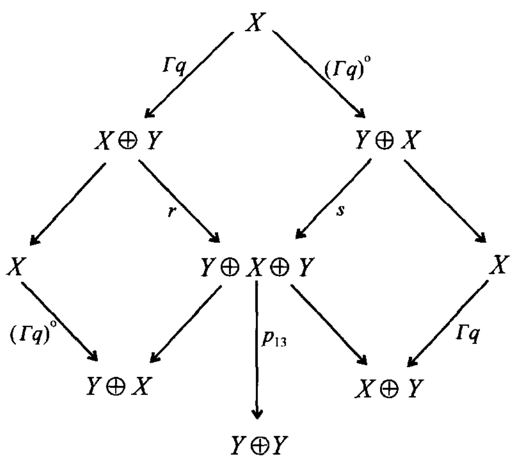
As for the splitting of reflexive relations, first observe that by the dual of the lemma in Section 3 of [3], reflexive relations split if and only if for each additive relation

$$X \overset{R}{\dashrightarrow} 0$$

there exists a *map* $M : X \dashrightarrow Y$ *in* Map($\mathbb{A}$Rel($\mathscr{P}$)) such that $MM^\circ = 1$ and $\top^\circ M = R$. Now, taking the *cokernel* $q : X \longrightarrow Y$ of a monomorphism $m : U \longrightarrow X$ which represents an additive relation R, we can still consider the graph Γq, but now defined by means of γ_{q1}, whose diagonal components are $(q, 1, q)$, as

$$\Gamma q = p_{23}\, \gamma_{q1}\, d_3 \,.$$

Γq is a map, since the composite with the first projection is the identity. The second equation is easy, and the first follows considering the diagram



where the unnamed arrows are the appropriate projections, and where the arrows r and s are constructed as follows: first observe that the kernel of $\gamma_{q1}\,(1 \oplus d)$ is $m' = \langle m, 0\rangle : U \longrightarrow X \oplus X$, which is uniquely determined since one component is 0; then, observe also that the cokernel of m' is the composite

$$c = p_{12}\, \gamma_{q1}\,(1 \oplus d) : X \oplus X \longrightarrow Y \oplus X$$

because is it epi, since q is, and has m' as kernel. Hence γ_{q1} factors uniquely through c as $s\,c$; the construction of r is similar. Since

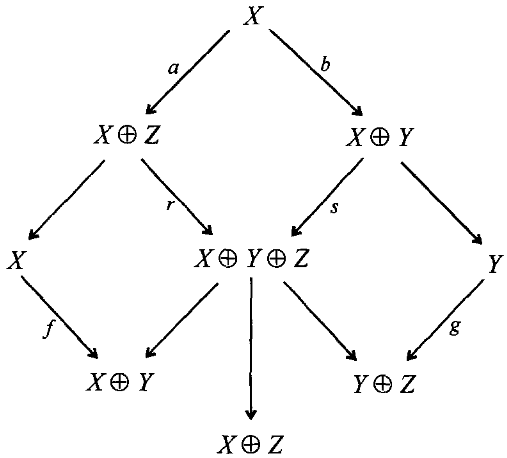
$$p_{12}\, s\, c = p_{12}\gamma_{q1}\,(1 \oplus d) = c\,,$$

we get $p_{12}\, s = 1$, and similarly $p_{23}\, r = 1$; because r and s have as diagonal components $((\Gamma q)^\circ, 1)$ and $(1, \Gamma q)$, respectively, the two bottom squares are pullbacks, and the whole diagram gives the composition $\Gamma q\,(\Gamma q)^\circ$ as:

$$\begin{aligned} \Gamma q\,(\Gamma q)^\circ &= Im\,(p_{13}\, s\,(\Gamma q)^\circ) = Im\,(p_{13}\, s\, c\, d) = Im\,(p_{13}\, \gamma_{q1}\,(1 \oplus d)\, d) \\ &= Im\,(p_{13}\, \gamma_{q1}\, d_3) = Im\,(d\, q) = d. \qquad \square \end{aligned}$$

5. The characterization theorem

Arrows $X \longrightarrow Y$ in $\mathbb{A}\mathscr{P}$ are precisely subobjects of $X \oplus Y$ in $\mathscr{P}$ whose first projection is an isomorphism; hence, observe that a *composition* in $\mathbb{A}\mathscr{P}$ is defined as follows: consider the diagram



where every square is a pullback and the unnamed arrows are the appropriate projections, and observe that the composite $p_{XY}\, s$ is an automorphism Θ such that $p_X\, \Theta = p_X$ and $p_Y\, \Theta = p_Y$, by point (iv) of Lemma 7; hence the arrow b is in fact $\Theta^{-1} f$, and the composite $p_X\, a$ is the identity; since the composite $p_{XZ}\, r$ is also an automorphism Ψ such that $p_X\, \Psi = p_X$ and $p_Z\, \Psi = p_Z$, the composition $g\, f$ in $\mathbb{A}\mathscr{P}$ turns out to be

$$g\, f = p_{XZ}\, s\, \Theta^{-1} f = p_{XZ}\, r\, a = \Psi a\,.$$

It is now easy to show that the composition with the second projection defines a *functor*

$$P : \mathbb{A}\mathscr{P} \longrightarrow P\,,$$

which is full and the identity on objects. The characterization theorem now follows from

Theorem 9. *When $\mathscr{P}$ is a projective category, $\mathscr{P}$ is canonically isomorphic to $\mathbb{P}\mathbb{A}\mathscr{P}$.*

Proof. First, observe that subobjects of an object Y in $\mathbb{A}\mathscr{P}$ are precisely subobjects of Y in $\mathscr{P}$; then, observe that inverse images of subobjects of Y in $\mathbb{A}\mathscr{P}$ along a map $f : X \longrightarrow X \oplus Y$ are calculated as inverse images in $\mathscr{P}$ of the projection $p_Y\, f$. In other words, the composition of P with the Grassmannian functor on $\mathscr{P}$ is the Grassmannian functor on $\mathbb{A}\mathscr{P}$. The result now follows: if the Grassmannian functor on $\mathscr{P}$ is faithful, then $\mathscr{P}$ is obviously isomorphic to $\mathbb{P}\mathbb{A}\mathscr{P}$. □

It only remains to investigate to what extent the structure of projective biproducts is ambiguous. Given two structures of projective biproducts on $\mathscr{P}$, say $\oplus$ and $\oplus'$, certainly the abelian categories $\mathbb{A}\mathscr{P}$ and $(\mathbb{A}\mathscr{P})'$ we construct out of them are in principle different, even though they have the same objects, since the construction depends from the structure of projective biproducts. However, once again point (iv) of Lemma 7 implies that for each pair of objects X_1, X_2, there are arrows commuting with projections (and injections)

$$A = A_{X_1,X_2} : X_1 \oplus' X_2 \longrightarrow X_1 \oplus X_2$$
$$B = B_{X_1,X_2} : X_1 \oplus X_2 \longrightarrow X_1 \oplus' X_2 ,$$

and hence such that BA is an automorphism of $X_1 \oplus' X_2$ such that $p_i'BA = p_i'$ and AB is an automorphism of $X_1 \oplus X_2$ such that $p_iBA = p_i$; it follows that A and B are *isomorphisms*. It follows that the composition with A_{X_1,X_2} induces an isomorphism

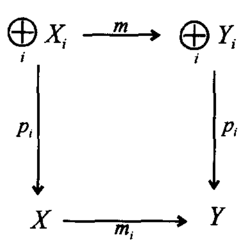
$$\mathbb{A}\mathscr{P}'(X_1,X_2) \simeq \mathbb{A}\mathscr{P}(X_1,X_2).$$

The point now is that we can*not* show that the correspondence, actually the *graph morphism* – which is the identity on objects – defined by composition with the isomorphisms $A = A_{(-,-)}$ does in fact preserve composition and identities; for, to show that composition is preserved, we would need at least to show that the correspondences defined for binary biproducts extend *naturally* to correspondences on ternary biproducts, which seems not to follow from the axioms; also, to show that identities are preserved, we would need to show *naturality* of the correspondences $A_{(-,-)}$ with respect to the diagonal in $\mathrm{Matr}(\mathscr{A}_0)$, which also seems not to follow from the axioms.

Looking for a natural isomorphism between two structures of projective biproducts, let us observe that point (iv) of Lemma 7 can be generalized to *finite families* of monomorphisms, in the following sense: given a finite family $(m_i : X_i \longrightarrow Y_i)_{i \in I}$ of monomorphisms, any arrow

$$m : \bigoplus_i X_i \longrightarrow \bigoplus_i Y_i$$

of diagonal components m_i has the property that the family of squares



is a joint pullback of m_i along the projections $p_i : \bigoplus_i Y_i \longrightarrow Y$. Of course, the dual statement for injections and epimorphisms holds too. In particular, one can show that the following property holds for projections and epimorphisms (and the dual for injections and monomorphisms):

"*given two arrows $H, K : U \longrightarrow \bigoplus_i X_i$ having the same components which are epimorphisms, there exists a unique endomorphism (in fact an automorphism) Θ of $\bigoplus_i X_i$ such that $\theta H = K$ and $p_i \Theta = p_i$*".

Observe that this property can be easily shown to hold in projective categories of the form $\mathbb{P}\mathscr{A}$, using the description of the Grassmannian equivalence relation given in Section 2. Also, observe that, given a finite family of objects $(X_i)_{i\in I}$, the endomorphisms Θ of $X = \bigoplus_i X_i$ such that $p_i\,\Theta = p_i$ are a subgroup $g(X)$ of the automorphisms group of X. However, even this observation seems not to imply that we can define a *natural* isomorphism between two structures of projective biproducts.

As a last remark, we should point out that the construction $\mathbb{P}\mathscr{A}$ does *not* extend to exact functors, already at the level of p-exact categories; it certainly does extend to exact functors which are *full on subobjects*. Also, if $\mathbb{A}$ and $\mathbb{B}$ are two abelian categories, then any equivalence between the associated projective categories which commutes, up to a natural isomorphism, with the canonical structures of projective biproducts, induces an equivalence between the two given abelian categories.

6. Affine categories

If $\mathscr{A}$ is an abelian category, or more generally an additive category with kernels, each comma category $\mathscr{E} = \mathscr{A}/X$, which is not additive anymore, unless $X = 0$, should be understood as an *affine category*, at least for the following two reasons: first, we can recover the additive category $\mathscr{A}$ as the category of pointed objects of $\mathscr{E}$; second, when $\mathscr{A}$ is the abelian category K-Vect of vector spaces, the comma category $\mathscr{E}=$ K-Vect/K is the category of (possibly empty) affine spaces (see [1] for a more detailed discussion and for a characterization of affine categories). From this point of view, each comma category is a category of affine spaces defined from $\mathscr{A}$, so that we can collect all of them in the *functor*

$$\mathrm{Aff} : \mathscr{A} \longrightarrow \mathrm{CAT}$$

defined on objects as

$$\mathrm{Aff}\, X = \mathscr{A}/X$$

and on arrows by composition, which we will call *the affine functor of* $\mathscr{A}$. When $\mathscr{A}$ is abelian, or p-exact only, the factorization gives for each X a functor

$$\mathrm{Im}_X : \mathrm{Aff}\, X \longrightarrow \mathscr{G}X$$

which is the X-component of a natural transformation from the affine functor to the Grassmannian one, i.e. for each arrow $f : X \longrightarrow Y$ the diagram

$$\begin{array}{ccc} \mathrm{Aff}\, X & \xrightarrow{\Sigma_f} & \mathrm{Aff}\, Y \\ {\scriptstyle \mathrm{Im}_X}\downarrow & & \downarrow{\scriptstyle \mathrm{Im}_Y} \\ \mathscr{G}X & \xrightarrow[f_{\bullet}]{} & \mathscr{G}Y \end{array}$$

commutes. Certainly, we can also ask about *the affine functor being 2-faithful*, meaning that "*when there exists a natural isomorphism $\Sigma_f \simeq \Sigma_g$, then $f = g$*", and convince ourselves that it is almost never the case, as for the Grassmannian functor. So, we can use the affine functor in the same way we did with the Grassmannian functor to define a new category

$$\mathbb{P}_{\mathrm{Aff}}\mathscr{A}$$

which we can call the "*affine-projective category of $\mathscr{A}$*", identifying two arrows f and g when there exists a natural isomorphism between Σ_f and Σ_g. Then, the natural questions arise to compare the two constructions $\mathbb{P}\mathscr{A}$ and $\mathbb{P}_{\mathrm{Aff}}\mathscr{A}$, and to characterize this last if the answer to the first question gives sense to such a problem.

First of all, observe that if two arrows of $\mathscr{A}$ are affine-equivalent, then they certainly are also Grassmannian-equivalent, so that there is a canonical comparison functor

$$C : \mathbb{P}_{\mathrm{Aff}}\mathscr{A} \longrightarrow \mathbb{P}\mathscr{A}$$

which is the identity on objects and full; one can check easily that in the case of vector spaces C is also faithful, and hence an isomorphism. But in the case of abelian groups, C is *not* faithful: indeed, it is easy to show that f is affine-equivalent to g if and only if they are equal or opposites (as a consequence of the fact that a group cannot be the set-theoretical union of two proper subgroups); and, because of the remark after Theorem 3, we know that the Grassmanian equivalence relation is strictly stronger than the equivalence relation using the invertibles of $\mathbf{Z}$. Thus the problem of characterizing affine-projective categories makes sense.

By repeating what we have done in the case of Grassmannian projective categories $\mathbb{P}\mathscr{A}$, one can find that categories of the form $\mathbb{P}_{\mathrm{Aff}}\mathscr{A}$ are certainly p-exact and have projective biproducts, so that recalling that *only these two properties are needed to reconstruct an abelian category*, one can hope that it would be possible to characterize affine-projective categories simply by substituting the requirement that the Grassmannian functor is faithful with the requirement that *the affine functor is 2-faithful*. However, the affine equivalence relation is:

"*Two arrows $f, g : X \longrightarrow Y$ are equivalent, when there exists a family of automorphisms $\alpha_x : U \longrightarrow U$ indexed by the arrows $x : U \longrightarrow X$ such that $g x \alpha_x = f x$ and $\alpha_x h = h \alpha_{xh}$, for each composable arrow h*".

Furthermore, passing to the quotient category $\mathbb{P}_{\mathrm{Aff}}\mathscr{A}$, we are not able to show that the corresponding equivalence relation is the identity relation, nor able to find an example where it is not. Clearly, if the answer to this question is negative, as we believe, the problem arises to characterize purely in terms of properties of abelian categories such as properties of classes of exact sequences or properties of lattices of subobjects, those abelian categories for which the answer is in fact positive, and in particular those rings for which the abelian category of modules has a *stable* affine equivalence relation. This class of abelian categories clearly contains the class of abelian categories for which the affine equivalence relation and the Grassmannian one do coincide, which is also a class that should be characterized.

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